Unconventional decay law for excited states in closed many-body systems

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We study the time evolution of an initially excited many-body state in a finite system of interacting Fermi particles in the situation when the interaction gives rise to the ''chaotic'' structure of compound states. This situation is generic for highly excited many-particle states in quantum systems such as heavy nuclei, complex atoms, quantum dots, spin systems, and quantum computers. For a strong interaction the leading term for the return probability $W(t)$ has the form $W(t)\simeq\exp(-\Delta_E^2 t^2)$ with Δ_E^2 as the variance of the strength function. The conventional exponential linear dependence $W(t)=C\exp(-\Gamma t)$ formally arises for a very larger time. However, the prefactor C turns out to be exponentially large, thus resulting in a strong difference from the conventional estimate for $W(t)$.

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It is known that highly excited states can be treated as ''chaotic'' ones in many-body systems, such as complex atoms [1], multicharged ions [2], nuclei [3] and spin systems [4,5], quantum computer models [6,7]. This happens due to a very high density of many-particle states that strongly increases with an increase of energy. For example, in the case of n Fermi particles occupying the finite number m of ''orbitals'' (single-particle states), the total number N of many-body states grows exponentially fast with an increase in number of particles, $N=m!/n!(m-n)!\sim\exp(c_0 n)$. Correspondingly, the density ρ_f of those many-body states that are directly coupled by a two-body interaction, also grows very fast. Therefore, even a relatively weak interaction between the particles can lead to a strong mixing between unperturbed many-body states (''basis states''). As a result, an exact (perturbed) eigenstate is represented by a chaotic superposition of a large number of components of basis states [8,9].

The number of principal basis components in such *chaotic* eigenstates can be estimated as $N_p\sim\Gamma/D$, where Γ is the *spreading width* of a typical component that can be estimated using the Fermi golden rule, and $D^{-1}(E)$ is the total density of many-body states. In the case of a quantum computer the interval between multiqubit energy levels $D\propto 1/N$ is extremely small, and practically it is impossible to resolve these levels. Moreover, both the temperature and finite time of computer operations lead to an energy uncertainty $\delta E \gg D$. A similar situation occurs for an electron that enters a many-electron quantum dot. In these cases the analysis of stationary chaotic eigenstates is not an adequate to real physical problems and one needs to consider the time evolution of wave functions. In this paper we extend the quantum chaos approach to the problem of time evolution of an initially excited basis state.

Exact many-body eigenstates $|k\rangle$ of the Hamiltonian $H=H_0+V$ of interacting Fermi particles can be expressed in terms of simple *shell-model basis states* $|f\rangle$ of H_0,

$$|k\rangle=\sum_f C_f^{(k)}|f\rangle;\quad |f\rangle=a_{f_1}^\dagger\cdots a_{f_n}^\dagger|0\rangle. \tag{1}$$

Here $|0\rangle$ is the ground state, $a_s^\dagger$ is the creation operator, and $C_f^{(k)}$ are components of an exact eigenstate in the unperturbed basis.

In application to quantum computer models the Hamiltonian H_0 describes a number of noninteracting *qubits* (two-level systems), and V stands for the interqubit interaction needed for a quantum computation (we assume time-independent V). In this case the basis state $|f\rangle$ is a product of single-qubit states, $a_s^\dagger$ is the spin-raising operator (if the ground state $|0\rangle$ corresponds to spins down), and chaotic eigenstates $|k\rangle$ are formed by the residual interaction V.

Below we consider the time evolution of the system, assuming that initially $(t=0)$ the system is in a specific basis state $|i\rangle$ (in the state with certain spins ''up'' for a quantum computer). This state can be expressed as a sum over exact eigenstates,

$$|i\rangle=\sum_k C_i^{(k)}|k\rangle, \tag{2}$$

therefore, the time-dependent wave function reads as

$$\Psi(t)=\sum_{k,f} C_i^{(k)}C_f^{(k)}|f\rangle\exp(-iE^{(k)}t). \tag{3}$$

Here $E^{(k)}$ are the eigenvalues corresponding to the eigenstates $|k\rangle$. The sum is taken over the eigenstates $|k\rangle$ and basis states $|f\rangle$ (in what follows, we put $\hbar=1$).

The probability $W_i=|A_i|^2=|\langle i|\Psi(t)\rangle|^2$ to find the system in the state $|i\rangle$ is determined by the amplitude

$$A_i=\langle i|\exp(-iHt)|i\rangle=\sum_k |C_i^{(k)}|^2\exp(-iE^{(k)}t)$$

$$\simeq\int P_i(E)\exp(-iEt)dE. \tag{4}$$

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Here we replaced the summation over a large number of eigenstates by the integration over their energies $E \equiv E^{(k)}$, and introduced the *strength function* (SF) $P_i(E)$ which is also known in the literature as the *local spectral density of states*,

$$P_i(E) \equiv \overline{|C_i^{(k)}|^2} \rho(E). \tag{5}$$

Here $\rho(E)$ is the density of states of the total Hamiltonian H, and the average is performed over a number of states with close energies.

In chaotic systems the strength function $P_i(E)$ is known to have the Breit-Wigner form for a relatively weak interaction, and is close to the Gaussian for a strong interaction [1,3]. Recently the following approximate general expression has been analytically found [10]:

$$P_i(E) = \frac{1}{2\pi} \frac{\Gamma_i(E)}{(E_i + \delta_i - E)^2 + \Gamma_i(E)^2/4}, \tag{6}$$

$$\Gamma_i(E) \simeq 2\pi \overline{|V_{if}|^2} \rho_f(E), \tag{7}$$

which is derived by making use of the approach described in Ref. [11]. Here $\Gamma_i(E)$ is some function of the total energy, δ_i is the correction to the unperturbed energy level E_i due to the residual interaction V_{if}, and $\rho_f(E)$ is the density of the basis states $|f\rangle$ directly connected with a given state $|i\rangle$ by the matrix elements V_{if}. The above result has been derived for the so-called two-body random interaction (TBRI) model [12] that describes n interacting Fermi-particles distributed over m orbitals, with the assumption that two-body matrix elements are completely random.

It is shown [10] that for a large number of particles the function $\Gamma_i(E)$ has the Gaussian form

$$\Gamma_i(E) \simeq 2\pi \Delta_E^2 \frac{1}{\sqrt{2\pi\sigma_f^2}} \exp\left(- \frac{(E - E_i - \omega_i)^2}{2\sigma_f^2} \right). \tag{8}$$

Here σ_f^2 is the variance of the density $\rho_f(E)$ (which also has the Gaussian form), and $E_i + \omega_i$ is the average energy of the basis states $|f\rangle$ directly connected with a given state $|i\rangle$ (if E_i is at the center of the spectrum we have $\omega_i = 0$). The width Δ_E of the SF is determined through the second moment, $\Delta_E^2 = \Sigma_{f \neq i} V_{if}^2$, which for the TBRI model is found to be [13]

$$\Delta_E^2 = \frac{1}{4} V_0^2 n(n-1)(m-n)(m-n+3), \tag{9}$$

with V_0^2 standing for the variance of the off-diagonal elements of a two-body interaction (Δ_E^2 for a quantum computer model is given in Ref. [7]).

If the interaction is not very strong we have $\sigma_f^2 \sim m^2 d_0^2/3$, where d_0 is the average distance between the single-particle energy levels (this estimate is valid for $\sigma_f \gg \Delta_E$). Therefore, for $m - n \gtrsim n \gg 1$ we have $\Gamma/\Delta_E \sim V_0 n/d_0$. This estimate may be compared with the criterium of chaos $V_0 \rho_f^{-1} \sim V_0 (m-n) n (n-1)/d_0 \sim V_0 m n^2/d_0 > 1$.

In the case of a relatively small (but nonperturbative) interaction (when $\Gamma_i \ll \Delta_E \ll \sigma_f$), the function $\Gamma(E)$ is very broad (i.e. it does not change significantly within the energy intervals $\sim \Gamma$ and Δ_E) and can be treated as a constant, $\Gamma(E) \simeq \Gamma_0$. In the case of a strong interaction, $\Gamma_0 \geqslant \Delta_E$, the dependence $\Gamma(E)$ in Eq. (6) is the leading one.

The knowledge of the strength function allows one to describe the dynamics of wave packets in the energy space. It is easy to find the evolution of $W_i(t)$ on a small time scale. Let us subtract the energy $E_i \equiv H_{ii}$ of the initial state in the exponent and make a second order expansion in $E - E_i$ in Eq. (4). This leads to the following result:

$$A_i = \exp(-iE_i t)(1 - \Delta_E^2 t^2/2) \tag{10}$$

and

$$W_i(t) = 1 - \Delta_E^2 t^2. \tag{11}$$

For a strong residual interaction, $\Gamma_0 \geqslant \Delta_E$, the return probabilty $W(t)$ turns out to be a function of $\Delta_E^2 t^2$ for a longer time [7]. Indeed, both the strength function and density of states in this limit are described by the Gaussian functions with the variance $\sigma^2 = \Delta_E^2$ (see details in Refs. [12–15]),

$$P_i(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[- \frac{(E - E_c)^2}{2\sigma^2} \right], \tag{12}$$

$$\rho(E) = \frac{N}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{E^2}{2\sigma^2} \right). \tag{13}$$

Here E_c is the center of the energy spectrum. Thus, Eq. (4) results in the Gaussian time dependence for $A_i(t)$ and $W_i(t)$,

$$A_i = \exp(-\Delta_E^2 t^2/2), \tag{14}$$

$$W_i(t) \simeq \exp(-\Delta_E^2 t^2). \tag{15}$$

Now let us consider large times. In this limit the result can be obtained by evaluation of the integral in Eq. (4) in the complex E plane. Specifically, one should close the contour of integration in the bottom part of the complex plane ($\mathrm{Im}\, E < 0$), in order to provide a vanishing contribution at infinity. Then, the large time limit is given by the pole of the strength function (6), closest to the real E axis. If Γ_i and δ_i in Eq. (5) do not depend on E, the integration gives the conventional exponential decay $W_i = \exp(-\Gamma t)$ [11]. However, the energy dependence of Γ is necessary to provide the finiteness of a second moment Δ_E^2 of the strength function. If $\Gamma < \Delta_E$, the closest pole is given by $\tilde{\Gamma} = -2\, \mathrm{Im}\, E_p$, where E_p is the solution of the equation $E_p = E_i + \delta_i(E_p) - i\Gamma(E_p)/2$ with a minimal imaginary part. If $\Gamma \ll \Delta_E$, then we have $\tilde{\Gamma} \approx \Gamma$. As a result, we obtain an exponential dependence for large t,

$$W_i(t) = C \exp(-\tilde{\Gamma} t), \tag{16}$$

with some constant C.

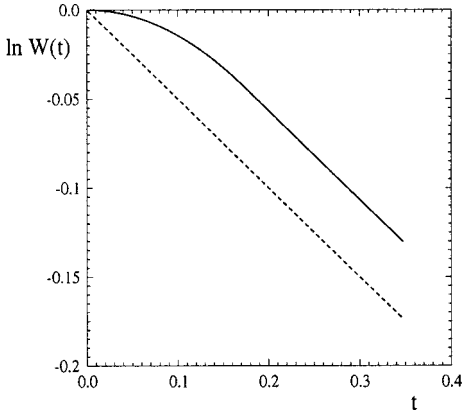

FIG. 1. Schematic time dependence $W(t)$ for $\tilde{\Gamma}=0.5$, $\Delta_E=1.2$; the dependence $W(t)=\exp(-\Delta_E^2 t^2)$ changes into $W(t)=\exp(-\Gamma t)$ at the point $t_c=\Gamma_p/\Delta_E^2\approx 0.17$.

It is important to describe $W_i(t)$ for arbitrary time t. Let us start from simple qualitative arguments. The transition between the Gaussian regime and simple exponential decay should occur near the time $t_c\sim\Gamma/\Delta_E^2$ where $\Delta_E^2 t_c^2\sim\Gamma t_c$. This gives the estimate $C\sim\exp(\Gamma^2/2\Delta_E^2)$ for the constant C. Indeed, for $t<t_c$ the quadratic exponential decay, $\exp(-\Delta_E^2 t^2)$, is slower than the linear one, $\exp(-\Gamma t)$. The matching of these two dependencies would naturally require the above expression for C. Thus, the constant C can be large if $\Gamma>\Delta_E$. The transition from one regime of the time dependence of $W_i(t)$ to another is schematically shown in Fig. 1.

In Ref. [7] the simple extrapolation formula for $W_i(t)$ has been suggested

$$W_i(t)=\exp\left(\frac{\Gamma^2}{2\Delta_E^2}-\sqrt{\frac{\Gamma^4}{4\Delta_E^4}+\Gamma^2 t^2}\right), \tag{17}$$

which interpolates (for $\Gamma<\Delta_E$) between the small (11) and large (16) time dependencies.

These qualitative results are supported by a more detailed consideration. Equations (6), (7), (8), and (12) provide us with an approximate formula for the strength function (tested by numerical calculations [16])

$$P(E)=B\frac{\exp\left[-\dfrac{(E-E_0)^2}{2\sigma^2}\right]}{(E-E_0)^2+\Gamma^2/4}, \tag{18}$$

which can be used, in conjunction with Eq.(4), to study the time dependence $W_i(t)$. Strictly speaking, this formula is valid near the center of the energy spectrum, otherwise one should take into account additional distortion effects.

Due to the normalization conditions, $\int P(E)dE=1$ and $\int E^2 P(E)dE=\Delta_E^2$, we have the following relations [16]:

$$\frac{1}{B}=2\left[1-\Phi\left(\frac{\Gamma}{\sigma\sqrt{8}}\right)\right]\frac{\pi}{\Gamma}\exp\left(\frac{\Gamma^2}{8\sigma^2}\right) \tag{19}$$

and

$$\Delta_E^2=B\left\{\sigma\sqrt{2\pi}-\frac{\pi\Gamma}{2}\exp\left(\frac{\Gamma^2}{8\sigma^2}\right)\left[1-\Phi\left(\frac{\Gamma}{\sigma\sqrt{8}}\right)\right]\right\}, \tag{20}$$

where $\Phi(z)$ is the error function.

The return probability $W(t)$ corresponding to the strength function (18) is then defined by the integral

$$A(t)=B\int_{-\infty}^{\infty}dE\,\frac{\exp\left(-\dfrac{E^2}{2\sigma^2}-iEt\right)}{(E-E_0)^2+\dfrac{\Gamma^2}{4}}. \tag{21}$$

The time dependencies of $A(t)$ and $W(t)$ for small time are given by Eqs. (14) and (15). If $\Gamma\ll\sigma$, the region of the applicability of these equations is very narrow. Indeed, in this case $\Delta_E^2\approx\sigma\Gamma/\sqrt{2\pi}$ and the condition $t\ll t_c\ll\Gamma/\Delta_E^2\sim 1/\sigma$ results in the relation $\Delta_E^2 t^2\ll 1$. The absolute value of the amplitude $A(t)$ in this case is given by the series in the parameter $(\sigma t)^2=(t/t_c)^2$,

$$|A(t)|=1-\frac{1}{2}\frac{\Gamma\sigma}{\sqrt{2\pi}}t^2+\frac{1}{24}\frac{\Gamma\sigma^3}{\sqrt{2\pi}}t^4+\cdots. \tag{22}$$

For large time, $t\gg t_c=1/\sigma$, the calculation of the integral in Eq. (20) leads to

$$W(t)\approx\exp\left(\frac{1}{\pi}\frac{\Gamma^2}{\Delta_E^2}-\Gamma t\right)$$

for the return probability. Here the correction $(1/\pi)(\Gamma^2/\Delta_E^2)\approx 2\Gamma/\sigma\sqrt{2\pi}$ is small. Indeed, the strength function in this case is close to the Lorentzian that gives a simple dependence $W(t)=\exp(-\Gamma t)$.

Another limit case of a large interaction, $\Gamma\gg\Delta_E$ (or, the same, $\Gamma\gg\sigma$), is more delicate. In this case the strength function is close to the Gaussian with $\Delta_E\approx\sigma$ and t_c is large, $t_c\sim\Gamma/\sigma^2\gg 1/\sigma$. The leading dependence of $W(t)$ in this case is the Gaussian, $W(t)\simeq\exp(-\Delta_E^2 t^2)$. Only for a long time $t\gg\Gamma/\sigma^2$ it becomes the simple exponential function

$$W(t)\approx\frac{\pi^2\Gamma^2}{8\Delta_E^2}\exp\left(\frac{1}{4}\frac{\Gamma^2}{\Delta_E^2}-\Gamma t\right). \tag{23}$$

It is important to note that even for a large time the return probability $W(t)$ has large correction factor $\exp[\frac{1}{4}\{\Gamma^2/(\Delta E)^2\}]$, in addition to the standard decay law $\exp(-\Gamma t)$.

Due to a finite number of particles, there are additional important features in the dynamics of wave packets, namely, the damped oscillations and the break of the decay for $W_i(t)$ [17]. The number of basis components $|f\rangle$ within the energy shell $|E_0-E_f|\leqslant\min(\Gamma,\sigma)\equiv\Delta$ is finite. Therefore, the decay stops if W_i is close to the equilibrium value defined as $W_\infty\equiv\overline{W_i(t\to\infty)}\approx 3N_{pc}^{-1}$. Here N_{pc} is the number of principal components in an eigenstate, $N_{pc}\sim\Delta/D$, where $D=\rho^{-1}$ is

the mean energy interval between all many-body levels. Note that the value of W_∞ is still at least three times larger than $W_f = N_{pc}^{-1}$ for any other component f, $\overline{W_i(\infty)} \geq \overline{W_f(\infty)}$ (see, details in Ref. [17]).

The equilibrium occurs because the average decay flux is equal to the average return flux. However, the return flux also leads to the damped oscillations of $W_i(t)$ and to the oscillations of a current number of the principal components $N_{pc}(t)$. These oscillations arise because the decay flux ''reflects'' from the edges of the energy shell when all components within this shell are populated. Period of these oscillations is about n_c/Δ where Δ is the inverse decay time, and n_c is the number of ''classes'' in the Hilbert space. This number can be defined as the number of interaction steps in the perturbative chain $H_{0\alpha_1} H_{\alpha_1\alpha_2} \cdots H_{\alpha_k\alpha_{n_c}}$ needed to populate all basis states within the energy shell. For example, in the TBRI model with 6 particles and 12 orbitals, the number of steps is $n_c \approx 3$ since each two-body interaction H_{ik} moves two particles to new orbitals.

In conclusion, we have studied generic features of the return probability $W(t)$ for a system to be found in an initially excited many-body state. Due to a two-body interaction between Fermi particles, the wave packet in the energy representation spreads over all basis states within the energy shell. The dependence $W(t)$ for small time is determined by a ballistic spread of the packet and is given by the expression (11). For large time, the decrease of $W(t)$ is determined by the form of the strength function $P(E)$. We have analyzed the behavior of $W(t)$ by making use of the analytical expression for $P(E)$, which is obtained for any strength of random two-body interaction between finite number of interacting Fermi particles.

We have shown that for the Breit-Wigner form of $P(E)$ (relatively weak interaction) the decay of $W(t)$ on a large time scale has the conventional exponential dependence, $W(t) \simeq \exp(-\Gamma t)$. On the other hand, for the Gaussian form of $P(E)$ (strong interaction) the time dependence $W(t)$ turns out to be of very specific. Namely, the leading term gives the quadratic exponential dependence, $W(t) \sim \exp(-\Delta_E^2 t^2)$, and only for a very large time the conventional exponential linear dependence formally recovers. However, in this case an additional prefactor C appears before the exponent, which turns out to be an exponentially large, thus resulting in a strong mortification of the standard exponential estimate for $W(t)$.

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